

Commutant of $\mathcal{L}_{\widehat{\mathfrak{sl}_2}}(4, 0)$ in the cyclic permutation orbifold of $\mathcal{L}_{\widehat{\mathfrak{sl}_2}}(1, 0)^{\otimes 4}$

Toshiyuki Abe¹

Faculty of Education, Ehime University
Matsuyama, Ehime 790-8577, Japan

and

Hiromichi Yamada²

Department of Mathematics, Hitotsubashi University
Kunitachi, Tokyo 186-8601, Japan

Abstract

We study the commutant of the vertex operator algebra $\mathcal{L}_{\widehat{\mathfrak{sl}_2}}(4, 0)$ in the cyclic permutation orbifold model $(\mathcal{L}_{\widehat{\mathfrak{sl}_2}}(1, 0)^{\otimes 4})^\tau$ with $\tau = (1\ 2\ 3\ 4)$. It is shown that the commutant is isomorphic to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -orbifold model of a tensor product of two lattice type vertex operator algebras of rank one.

1 Introduction

Let $A_1 = \mathbb{Z}\alpha$, $\langle \alpha, \alpha \rangle = 2$ be a root lattice of type A_1 . It is well-known that the vertex operator algebra V_{A_1} associated to the lattice A_1 is isomorphic to a simple affine vertex operator algebra $\mathcal{L}_{\widehat{\mathfrak{sl}_2}}(1, 0)$ of type \mathfrak{sl}_2 with level 1. For an integer $k \geq 2$, the cyclic sums of the weight one vectors in V_{A_1} in the tensor product $V_{A_1}^{\otimes k}$ of k copies of V_{A_1} generate a vertex operator subalgebra isomorphic to $\mathcal{L}_{\widehat{\mathfrak{sl}_2}}(k, 0)$. The commutant M of $\mathcal{L}_{\widehat{\mathfrak{sl}_2}}(k, 0)$ in $V_{A_1}^{\otimes k}$ has been studied well (see for example [JL], [LS], [LY]). Among other things the classification of irreducible modules for M and the rationality of M were established in [JL].

Let τ be a cyclic permutation on the tensor components of $V_{A_1}^{\otimes k}$ of length k . Then τ is an automorphism of the vertex operator algebra $V_{A_1}^{\otimes k}$ and every element of $\mathcal{L}_{\widehat{\mathfrak{sl}_2}}(k, 0)$ is fixed by τ . Thus τ induces an automorphism of M . Our main concern is the orbifold model M^τ of M by τ , that is, the set of fixed points of τ in M , which is the commutant of $\mathcal{L}_{\widehat{\mathfrak{sl}_2}}(k, 0)$ in the orbifold model $(V_{A_1}^{\otimes k})^\tau$.

The vertex operator algebra $\mathcal{L}_{\widehat{\mathfrak{sl}_2}}(k, 0)$ contains a subalgebra T isomorphic to a vertex operator algebra associated to a rank one lattice generated by a square norm $2k$ element, which corresponds to a Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$. The commutant of T in $V_{A_1}^{\otimes k}$ is a lattice type vertex operator algebra $V_{\sqrt{2}A_{k-1}}$.

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On the other hand the commutant $K(\mathfrak{sl}_2, k)$ of T in $\mathcal{L}_{\widehat{\mathfrak{sl}_2}}(k, 0)$ is called a parafermion vertex operator algebra of type \mathfrak{sl}_2 . The parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ has been studied both in mathematics and in physics from various points of view (see for example [ALY], [DLY3], [DLWY], [DL]). We note that $M \otimes \mathcal{L}_{\widehat{\mathfrak{sl}_2}}(k, 0) \subset V_{A_1}^{\otimes k}$ and $M \otimes K(\mathfrak{sl}_2, k) \subset V_{\sqrt{2}A_{k-1}}$. In fact, M is the commutant of $K(\mathfrak{sl}_2, k)$ in $V_{\sqrt{2}A_{k-1}}$.

If $k = 2$, then M is isomorphic to the simple Virasoro vertex operator algebra $L(\frac{1}{2}, 0)$ of central charge $\frac{1}{2}$. In this case τ acts trivially on M and M^τ coincides with M . The first nontrivial case, that is, the orbifold model M^τ for the case $k = 3$ was studied in [DLTY]. It was shown that M^τ is a W_3 -algebra of central charge $\frac{6}{5}$. Furthermore, the classification of irreducible modules for M^τ was obtained and their properties were discussed in detail. Those results were used for the study of the vertex operator algebra $(V_{\sqrt{2}A_2})^\tau$ in [TY].

In this paper we consider the orbifold model M^τ for the case $k = 4$. The study of M^τ should lead to a better understanding of the structure of $(V_{\sqrt{2}A_3})^\tau$, for $M^\tau \otimes K(\mathfrak{sl}_2, 4)$ is contained in $(V_{\sqrt{2}A_3})^\tau$.

Let $L = \mathbb{Z}\alpha \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\alpha$ be an orthogonal sum of three copies of $\mathbb{Z}\alpha$, where $\langle \alpha, \alpha \rangle = 2$. The main idea is the use of an automorphism ρ of the vertex operator algebra V_L studied in [DLY2], which maps V_N onto V_L^+ . Here N is a sublattice of L isomorphic to the sublattice $\sqrt{2}A_3$ of A_1^4 . It was shown in [DLY2] that $\rho(M) = \text{Com}_{V_L^+}(V_{\mathbb{Z}\gamma})$; the commutant of $V_{\mathbb{Z}\gamma}$ in V_L^+ , where $\gamma = (\alpha, \alpha, \alpha) \in L$.

The cyclic permutation τ on the tensor components of $V_{A_1}^{\otimes 4} = V_{A_1^4}$ is a lift of an isometry of the underlying lattice A_1^4 . We denote the isometry of A_1^4 by the same symbol τ . The sublattice $\sqrt{2}A_3$ of A_1^4 is invariant under τ . Hence we can discuss an isometry $\tilde{\tau}$ of N corresponding to the isometry τ of $\sqrt{2}A_3$ by the isomorphism $N \cong \sqrt{2}A_3$. We extend $\tilde{\tau}$ to an isometry of L and consider its lift to an automorphism of the vertex operator algebra V_L . We denote the automorphism by the same symbol $\tilde{\tau}$. Let $\tau' = \rho\tilde{\tau}\rho^{-1}$ be the conjugate of $\tilde{\tau}$ by ρ so that $\rho(M^\tau) = \text{Com}_{V_L^+}(V_{\mathbb{Z}\gamma})^{\tau'}$. It turns out that $\text{Com}_{V_L^+}(V_{\mathbb{Z}\gamma})^{\tau'}$ can be expressed as $(V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2})^G$, where γ_1 and γ_2 are elements of L of square norm 12 and 4, respectively and G is a group of automorphisms of $V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

It is known that the vertex operator algebra M is generated by the set of conformal vectors ω_α of central charge $\frac{1}{2}$ associated to the positive roots α of type A_3 (see [JL], [LS]). However, it is difficult to describe the properties of the orbifold model M^τ in terms of those generators of M . By the result in this paper we can discuss $(V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2})^G$ instead of M^τ , which seems to be easy to treat.

This paper is organized as follows. In Section 2 we review basic materials of vertex operator algebras such as conformal vectors and the commutant of a vertex operator subalgebra. In Section 3 we discuss two kinds of automor-

phisms of a vertex operator algebra V_L associated to a positive definite even lattice L , one is a lift of the -1 -isometry of the lattice L and the other is an exponential of the operator $h_{(0)}$ for $h \in \mathbb{C} \otimes_{\mathbb{Z}} L$. We also recall three automorphisms of a rank one lattice type vertex operator algebra $V_{\mathbb{Z}\alpha}$ studied in [DLY1], [DLY2] for $\langle \alpha, \alpha \rangle = 2$. In Section 4 we introduce the commutant M of $\mathcal{L}_{\widehat{\mathfrak{sl}_2}}(4, 0)$ in $\mathcal{L}_{\widehat{\mathfrak{sl}_2}}(1, 0)^{\otimes 4}$ and its orbifold model M^τ by τ . Finally, in Section 5 we prove that M^τ is isomorphic to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -orbifold model of a tensor product of two rank one lattice type vertex operator algebras.

The automorphism ρ of the vertex operator algebra V_L plays a key role in our argument. The use of ρ was suggested by Ching Hung Lam. The authors are grateful to him for the important advice.

2 Preliminaries

In this section we review some basic notions and notations for vertex operator algebras (see [MN], [K], [LL]). Let $V = (V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra with the vacuum vector $\mathbf{1}$ and the Virasoro vector ω . We denote ω by ω^V also. The n -th product of $u, v \in V$ will be written as $u_{(n)}v$ for $n \in \mathbb{Z}$. We often regard $u_{(n)}$ as a \mathbb{C} -linear endomorphism of V . Two vectors u and v in V are said to be *mutually commutative* if $u_{(n)}v = 0$ for all $n \in \mathbb{Z}_{\geq 0}$. The eigenspace V_n for $L_0 = \omega_{(1)}^V$ of eigenvalue $n \in \mathbb{Z}$ is finite dimensional. A vector in V_n is said to be of weight n .

A vertex operator subalgebra of V is a vertex subalgebra U equipped with a Virasoro vector ω^U . When $\omega^V = \omega^U$, U is said to be full. For a pair of a vertex operator algebra V and its subalgebra U , the subspace

$$\text{Com}_V(U) = \{v \in V \mid u_{(n)}v = 0 \text{ for } n \in \mathbb{Z}_{\geq 0} \text{ and } u \in U\}$$

becomes a vertex operator algebra with Virasoro vector $\omega^{\text{Com}_V(U)} = \omega^V - \omega^U$. We call it the *commutant* of U in V . Actually, it is known that

$$\text{Com}_V(U) = \{v \in V \mid \omega_{(0)}^U v = 0\} \quad (2.1)$$

(see [FZ, Theorem 5.2]). Hence the commutant of U in V depends only on the Virasoro vector of U .

A vector $e \in V_2$ is called a *conformal vector* if $L_n^e = e_{(n+1)}$, $n \in \mathbb{Z}$ give a representation for the Virasoro algebra on V of certain central charge. The Virasoro vector of a vertex operator subalgebra U of V is a conformal vector of V . Let e be a conformal vector in V . For any vertex operator subalgebra U with $\omega^U = e$, the commutant $\text{Com}_V(U)$ does not depend on U by (2.1). In such a case we may write $\text{Com}_V(e) = \text{Com}_V(U)$.

Proposition 2.1. *Let V be a vertex operator algebra and e^1, e^2 mutually commutative conformal vectors in V . Then $\text{Com}_V(e^1 + e^2) = \text{Com}_{\text{Com}_V(e^1)}(e^2)$.*

Proof. Let U be a vertex subalgebra generated by e^1 and e^2 . Since e^1 and e^2 are mutually commutative, $e_1 + e_2$ is the Virasoro vector of U . Thus we have

$$\begin{aligned}\text{Com}_V(e^1 + e^2) &= \text{Com}_V(U) \\ &= \{v \in V \mid e_{(n)}^1 v = e_{(n)}^2 v = 0 \text{ for } n \in \mathbb{Z}_{\geq 0}\} \\ &= \{v \in \text{Com}_V(e^1) \mid e_{(n)}^2 v = 0 \text{ for } n \in \mathbb{Z}_{\geq 0}\} \\ &= \text{Com}_{\text{Com}_V(e^1)}(e^2).\end{aligned}$$

□

An automorphism of a vertex operator algebra V is a linear isomorphism of V preserving all n -th product, and fixing the vacuum vector $\mathbf{1}$ and the Virasoro vector ω^V . For a group G consisting of automorphisms of V , the subset

$$V^G = \{v \in V \mid g(v) = v \text{ for } g \in G\}$$

is a full vertex operator subalgebra of V , which is called the orbifold model of V by G . When $G = \langle \tau \rangle$ is a cyclic group, we denote V^G by V^τ simply.

Let V be a vertex operator algebra and G an automorphism group of V . Let U a vertex operator subalgebra of V and assume that $g(\omega^U) = \omega^U$ for any $g \in G$. Then the restriction g' of $g \in G$ to $\text{Com}_V(U)$ gives rise to an automorphism of $\text{Com}_V(U)$. In fact, for the automorphism group $H = \{g' \mid g \in G\}$ of $\text{Com}_V(U)$, we have

$$\text{Com}_V(U)^H = \text{Com}_{V^G}(\omega^U). \quad (2.2)$$

3 Lattice type vertex operator algebras and their automorphisms

In this section we discuss certain automorphisms of lattice type vertex operator algebras. Let V_L be the vertex operator algebra constructed in [FLM] for a positive definite even lattice (L, \langle, \rangle) of rank d . As a vector space V_L is isomorphic to a tensor product of the symmetric algebra $S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}])$ and the twisted group algebra $\mathbb{C}\{L\}$ of L , where $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$. In this paper we only consider the case where $\langle \alpha, \alpha \rangle \in 4\mathbb{Z}$ for any $\alpha \in L$ or L is an orthogonal sum of rank one lattices. In such a case the central extension \hat{L} of L studied in [FLM] splits and $\mathbb{C}\{L\}$ is canonically isomorphic to the ordinary group algebra $\mathbb{C}[L]$. Thus we take $\mathbb{C}[L]$ in place of $\mathbb{C}\{L\}$ here. A standard basis of $\mathbb{C}[L]$ is denoted by $\{e^\alpha \mid \alpha \in L\}$ with multiplication $e^\alpha e^\beta = e^{\alpha+\beta}$. The vacuum vector of V_L is $\mathbf{1} = 1 \otimes e^0$, and the Virasoro vector is given by

$$\omega^{V_L} = \frac{1}{2} \sum_{i=1}^d (h_i \otimes t^{-1})^2 \otimes e^0,$$

where $\{h_1, \dots, h_d\}$ is an orthonormal basis of \mathfrak{h} . Every eigenvalue for $L_0 = \omega_{(1)}^{V_L}$ on V_L is a nonnegative integer and the eigenspace $(V_L)_n$ with eigenvalue n is finite dimensional.

For simplicity, we regard the sets L , \mathfrak{h} and $\{e^\alpha | \alpha \in L\}$ as subsets of V_L , respectively, under the identification

$$\alpha = (\alpha \otimes 1)_{(-1)} \mathbf{1}, \quad h = h_{(-1)} \mathbf{1}, \quad e^\alpha = 1 \otimes e^\alpha$$

for $\alpha \in L$ and $h \in \mathfrak{h}$. Then we have $(V_L)_0 = \mathbb{C} \mathbf{1}$ and

$$(V_L)_1 = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in L \\ \langle \alpha, \alpha \rangle = 2}} \mathbb{C} e^\alpha.$$

In fact, the weight of e^α is $\frac{1}{2} \langle \alpha, \alpha \rangle$ for $\alpha \in L$.

We will need two kinds of automorphisms of V_L . One is an involution θ_L given by a lift of the -1 -isometry of the lattice L . We have

$$\theta_L(\beta) = -\beta, \quad \theta_L(e^\beta) = e^{-\beta}$$

for $\beta \in L$. The set $(V_L)^{\theta_L}$ of fixed points of θ_L is also denoted by V_L^+ . The other is an inner automorphism $I_h = \exp(2\pi\sqrt{-1}h_{(0)})$ for $h \in \mathfrak{h}$. We have

$$I_h(\beta) = \beta, \quad I_h(e^\beta) = e^{2\pi\sqrt{-1}\langle h, \beta \rangle} e^\beta$$

for $\beta \in L$. In particular,

$$I_{\alpha/2\langle \alpha, \alpha \rangle}(e^{m\alpha}) = (-1)^m e^{m\alpha}$$

for $m \in \mathbb{Z}$ and $0 \neq \alpha \in L$. Since $I_{h_1} I_{h_2} = I_{h_1+h_2}$ for $h_1, h_2 \in \mathfrak{h}$, the automorphism I_h is of finite order if and only if $h \in \frac{1}{T}L$ for some $T \in \mathbb{Z}_{>0}$. By the definition of θ_L and I_h , we see that

$$\theta_L I_h \theta_L = I_{-h}$$

for any $h \in \mathfrak{h}$. Therefore,

$$I_{-\frac{h}{2}}(I_h \theta_L) I_{\frac{h}{2}} = \theta_L. \quad (3.1)$$

That is, $I_h \theta_L$ is conjugate to θ_L in $\text{Aut}(V_L)$.

Let $\mathbb{Z}\alpha$ be a rank one lattice with $\langle \alpha, \alpha \rangle = 2$, which is the root lattice of type A_1 . In [DLY1, Section 2], three involutions θ_1 , θ_2 and σ of $V_{\mathbb{Z}\alpha}$ are considered. The involutions θ_1 and θ_2 are expressed as

$$\theta_1 = I_{\frac{\alpha}{4}}, \quad \theta_2 = \theta_{\mathbb{Z}\alpha},$$

and σ is a unique extension of the involution of the Lie algebra $(V_{\mathbb{Z}\alpha})_1 \cong \mathfrak{sl}_2(\mathbb{C})$ given by

$$\sigma(\alpha) = E^\alpha, \quad \sigma(E^\alpha) = \alpha, \quad \sigma(F^\alpha) = -F^\alpha, \quad (3.2)$$

where $E^\alpha = e^\alpha + e^{-\alpha}$ and $F^\alpha = e^\alpha - e^{-\alpha}$. As automorphisms of $V_{\mathbb{Z}\alpha}$, we have

$$\sigma \theta_{\mathbb{Z}\alpha} \sigma = I_{\frac{\alpha}{4}}. \quad (3.3)$$

4 Orbifold model M^τ

In this section we introduce an orbifold model M^τ . We use the notation X_N to denote the root lattice of type X_N . We also write X_N^i for an orthogonal sum of i copies of the root lattice X_N .

For simplicity, we write $\mathcal{L}(k, 0)$ for the simple affine vertex operator algebra $\mathcal{L}_{\widehat{\mathfrak{sl}_2}}(k, 0)$ associated to $\mathfrak{sl}_2(\mathbb{C})$ of positive integer level k . It is well-known that $\mathcal{L}(1, 0)$ is isomorphic to the lattice type vertex operator algebra V_{A_1} associated to the root lattice of type A_1 . Thus we have natural isomorphisms

$$\mathcal{L}(1, 0)^{\otimes 4} \cong V_{A_1}^{\otimes 4} \cong V_{A_1^4} \quad (4.1)$$

of vertex operator algebras. Let τ be a cyclic permutation on $\mathcal{L}(1, 0)^{\otimes 4}$ defined by

$$\tau(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = a_4 \otimes a_1 \otimes a_2 \otimes a_3$$

for $a_i \in \mathcal{L}(1, 0)$. In fact, τ is a lift of an isometry

$$\tau : \alpha_i \mapsto \alpha_{i+1}, \quad i \in \mathbb{Z}/4\mathbb{Z} \quad (4.2)$$

of the lattice

$$A_1^4 = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \oplus \mathbb{Z}\alpha_4 \quad (4.3)$$

with $\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j}$.

Set

$$H = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$$

and consider a sublattice $\sqrt{2}A_3 = \sum_{i,j=1}^4 \mathbb{Z}(\alpha_i - \alpha_j)$ of A_1^4 . We note that $\mathbb{Z}H$ and $\sqrt{2}A_3$ are mutually orthogonal and that $H \equiv 4\alpha_1$ modulo $\sqrt{2}A_3$. Hence $A_1^4 = \cup_{i=0}^3 (i\alpha_1 + \mathbb{Z}H + \sqrt{2}A_3)$ and

$$V_{A_1^4} = \oplus_{i=0}^3 V_{i\alpha_1 + \mathbb{Z}H + \sqrt{2}A_3}$$

as $V_{\mathbb{Z}H + \sqrt{2}A_3}$ -modules. It follows that

$$\text{Com}_{V_{A_1^4}}(V_{\mathbb{Z}H}) = V_{\sqrt{2}A_3}.$$

Since τ leaves $V_{\mathbb{Z}H}$ invariant, τ induces an automorphism of the vertex operator algebra $V_{\sqrt{2}A_3}$, which will be also denoted by τ . This automorphism is a lift of the restriction of the isometry τ (4.2) of A_1^4 to its sublattice $\sqrt{2}A_3$:

$$\tau(\alpha_i - \alpha_{i+1}) = \alpha_{i+1} - \alpha_{i+2}, \quad i \in \mathbb{Z}/4\mathbb{Z}. \quad (4.4)$$

Since $V_{\mathbb{Z}H}$ is contained in $(V_{A_1^4})^\tau$, it follows from (2.2) that

$$(V_{\sqrt{2}A_3})^\tau = \text{Com}_{(V_{A_1^4})^\tau}(V_{\mathbb{Z}H})$$

We next take two vectors

$$E = e^{\alpha_1} + e^{\alpha_2} + e^{\alpha_3} + e^{\alpha_4}, \quad F = e^{-\alpha_1} + e^{-\alpha_2} + e^{-\alpha_3} + e^{-\alpha_4}$$

in $(V_{A_1^4})_1^\tau$. Then the set $\{E, H, F\}$ generates a vertex operator subalgebra U of $(V_{A_1^4})^\tau$ isomorphic to $\mathcal{L}(4, 0)$. We consider the commutant

$$M = \text{Com}_{V_{A_1^4}}(U) \cong \text{Com}_{\mathcal{L}(1,0)^{\otimes 4}}(\mathcal{L}(4, 0)).$$

We note that U contains $V_{\mathbb{Z}H}$ as a vertex operator subalgebra. The commutant

$$K_0 = \text{Com}_U(V_{\mathbb{Z}H})$$

has been studied in [ALY], [DLY3] and [DLWY]. The Virasoro vector of K_0 is $\omega^{K_0} = \omega^U - \omega^{V_{\mathbb{Z}H}}$. Since ω^{K_0} and $\omega^{V_{\mathbb{Z}H}}$ are mutually commutative conformal vectors and since $\text{Com}_{V_{A_1^4}}(\omega^{V_{\mathbb{Z}H}}) = V_{\sqrt{2}A_3}$, we see that

$$M = \text{Com}_{V_{A_1^4}}(\omega^U) = \text{Com}_{V_{\sqrt{2}A_3}}(\omega^{K_0}) \quad (4.5)$$

by Proposition 2.1. Since U is contained in $(V_{A_1^4})^\tau$, we have

$$M^\tau = \text{Com}_{(V_{\sqrt{2}A_3})^\tau}(\omega^{K_0}) = \left(\text{Com}_{V_{\sqrt{2}A_3}}(\omega^{K_0}) \right)^\tau. \quad (4.6)$$

5 Main results

In this section we show that M^τ is isomorphic to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -orbifold model of a tensor product of two lattice type vertex operator algebras of rank one. Following [DLY2], we study an isomorphism between the vertex operator algebras $V_{\sqrt{2}A_3}$ and $V_{A_1^3}^+$. We remark that another isomorphism was considered in [DLY1] (see [DLY2, Remark 3.2]).

Throughout this section, let

$$L = \mathbb{Z}\alpha \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\alpha$$

be an orthogonal sum of three copies of $\mathbb{Z}\alpha$, where $\langle \alpha, \alpha \rangle = 2$. Set

$$\alpha^{(1)} = (\alpha, 0, 0), \quad \alpha^{(2)} = (0, \alpha, 0), \quad \alpha^{(3)} = (0, 0, \alpha),$$

so that $L = \mathbb{Z}\alpha^{(1)} + \mathbb{Z}\alpha^{(2)} + \mathbb{Z}\alpha^{(3)}$ and $\langle \alpha^{(i)}, \alpha^{(j)} \rangle = 2\delta_{i,j}$. The vertex operator algebra V_L is isomorphic to the tensor product $V_{A_1}^{\otimes 3}$ of three copies of $V_{A_1} = V_{\mathbb{Z}\alpha}$. Hence the involution σ of $V_{\mathbb{Z}\alpha}$ defined in (3.2) induces naturally an involution $\sigma \otimes \sigma \otimes \sigma$ of V_L . Let

$$\rho = \text{I}_{\frac{1}{4}(\alpha^{(2)} + \alpha^{(3)})}(\sigma \otimes \sigma \otimes \sigma) \in \text{Aut}(V_L) \quad (5.1)$$

be a composite of $\sigma \otimes \sigma \otimes \sigma$ and the inner automorphism $I_{\frac{1}{4}(\alpha^{(2)} + \alpha^{(3)})}$ of V_L with respect to $\frac{1}{4}(\alpha^{(2)} + \alpha^{(3)}) \in \frac{1}{4}L$ (see [DLY2, Section 3]). We note that $I_{\frac{1}{4}(\alpha^{(2)} + \alpha^{(3)})} = 1 \otimes I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha}$. By the definition of ρ , we have

$$\rho(\alpha^{(1)}) = E^{\alpha^{(1)}}, \quad \rho(\alpha^{(i)}) = -E^{\alpha^{(i)}} \quad (i = 2, 3), \quad (5.2)$$

$$\rho(E^{\alpha^{(i)}}) = \alpha_i \quad (i = 1, 2, 3), \quad (5.3)$$

$$\rho(F^{\alpha^{(1)}}) = -F^{\alpha^{(1)}}, \quad \rho(F^{\alpha^{(i)}}) = F^{\alpha_i} \quad (i = 2, 3). \quad (5.4)$$

Set

$$\begin{aligned} \beta_1 &= \alpha^{(1)} + \alpha^{(2)}, & \beta_2 &= -\alpha^{(2)} + \alpha^{(3)}, & \beta_3 &= -\alpha^{(1)} + \alpha^{(2)}, \\ \gamma &= \alpha^{(1)} + \alpha^{(2)} + \alpha^{(3)}. \end{aligned}$$

Then $\{\frac{1}{\sqrt{2}}\beta_1, \frac{1}{\sqrt{2}}\beta_2, \frac{1}{\sqrt{2}}\beta_3\}$ forms the set of simple roots of type A_3 . We consider the sublattice

$$N = \sum_{i,j=1}^3 \mathbb{Z}(\alpha^{(i)} \pm \alpha^{(j)}) = \langle \beta_1, \beta_2, \beta_3 \rangle_{\mathbb{Z}}$$

of L . It is known that $N \cong \sqrt{2}D_3$ is isomorphic to the sublattice $\sqrt{2}A_3$ of the lattice A_1^4 (4.3) discussed in Section 4 by the correspondence

$$\beta_1 \leftrightarrow \alpha_1 - \alpha_2, \quad \beta_2 \leftrightarrow \alpha_2 - \alpha_3, \quad \beta_3 \leftrightarrow \alpha_3 - \alpha_4. \quad (5.5)$$

This isomorphism between the lattices N and $\sqrt{2}A_3$ induces an isomorphism between the vertex operator algebras V_N and $V_{\sqrt{2}A_3}$. Thus we can think of the vertex operator subalgebras M and K_0 of $V_{\sqrt{2}A_3}$ discussed in Section 4 as vertex operator subalgebras of V_N .

We need the following facts in [DLY2].

Theorem 5.1. (1) $\rho(V_N) = V_L^+$.

(2) $\rho(\omega^{K_0}) = \omega^{V_{\mathbb{Z}\gamma}}$.

(3) $\rho(M) = \text{Com}_{V_L^+}(\omega^{V_{\mathbb{Z}\gamma}})$.

Proof. The assertions (1) and (2) follow from [DLY2, Lemmas 3.4 (3) and 3.3 (3)]. Then the assertion (3) follows from (4.5). \square

Under the isomorphism (5.5) between N and $\sqrt{2}A_3$, the isometry τ (4.4) of the lattice $\sqrt{2}A_3$ corresponds to an isometry

$$\beta_1 \mapsto \beta_2 \mapsto \beta_3 \mapsto -\alpha^{(2)} - \alpha^{(3)} \mapsto \beta_1$$

of the lattice N . This isometry of N is the restriction of an isometry $\tilde{\tau}$ of the lattice L of order 4 given by

$$\tilde{\tau} : \alpha^{(1)} \mapsto \alpha^{(3)} \mapsto -\alpha^{(1)} \mapsto -\alpha^{(3)} \mapsto \alpha^{(1)}, \quad \alpha^{(2)} \leftrightarrow -\alpha^{(2)}.$$

The isometry $\tilde{\tau}$ of L lifts to an automorphism of the vertex operator algebra V_L of order 4, which is also denoted by $\tilde{\tau}$.

Actually,

$$\tilde{\tau} = (\theta_{\mathbb{Z}\alpha} \otimes \theta_{\mathbb{Z}\alpha} \otimes 1)t_{13}$$

is a composite of t_{13} and $\theta_{\mathbb{Z}\alpha} \otimes \theta_{\mathbb{Z}\alpha} \otimes 1$, where t_{13} denotes the transposition of the first component and the third one of the tensor product $V_L = V_{A_1}^{\otimes 3}$. By direct calculations, we have

$$\tilde{\tau}(\alpha^{(1)}) = \alpha^{(3)}, \quad \tilde{\tau}(\alpha^{(2)}) = -\alpha^{(2)}, \quad \tilde{\tau}(\alpha^{(3)}) = -\alpha^{(1)}, \quad (5.6)$$

$$\tilde{\tau}(e^{\pm\alpha^{(1)}}) = e^{\pm\alpha^{(3)}}, \quad \tilde{\tau}(e^{\pm\alpha^{(2)}}) = e^{\mp\alpha^{(2)}}, \quad \tilde{\tau}(e^{\pm\alpha^{(3)}}) = e^{\mp\alpha^{(1)}}. \quad (5.7)$$

Now we consider the conjugate τ' of $\tilde{\tau}$ by ρ (5.1).

$$\tau' = \rho\tilde{\tau}\rho^{-1} \in \text{Aut}(V_L).$$

Lemma 5.2. *As automorphisms of V_L , we have*

$$\tau' = (1 \otimes I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha})t_{13}. \quad (5.8)$$

Proof. Using (3.3), we can calculate as follows.

$$\begin{aligned} \tau' &= \rho\tilde{\tau}\rho^{-1} \\ &= I_{\frac{1}{4}(\alpha_2+\alpha_3)}\sigma\tilde{\tau}\sigma I_{\frac{1}{4}(\alpha_2+\alpha_3)} \\ &= (1 \otimes I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha})(\sigma \otimes \sigma \otimes \sigma)(\theta_{\mathbb{Z}\alpha} \otimes \theta_{\mathbb{Z}\alpha} \otimes 1)t_{13} \\ &\quad \circ (\sigma \otimes \sigma \otimes \sigma)(1 \otimes I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha}) \\ &= (\sigma\theta_{\mathbb{Z}\alpha} \otimes I_{\frac{1}{4}\alpha}\sigma\theta_{\mathbb{Z}\alpha} \otimes I_{\frac{1}{4}\alpha}\sigma)t_{13}(\sigma \otimes \sigma I_{\frac{1}{4}\alpha} \otimes \sigma I_{\frac{1}{4}\alpha}) \\ &= (\sigma\theta_{\mathbb{Z}\alpha}\sigma I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha}\sigma\theta_{\mathbb{Z}\alpha}\sigma I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha})t_{13} \\ &= (1 \otimes I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha})t_{13}. \end{aligned}$$

□

Recall that $\gamma = \alpha^{(1)} + \alpha^{(2)} + \alpha^{(3)}$. We set

$$\gamma_1 = \beta_2 - \beta_3 = \alpha^{(1)} - 2\alpha^{(2)} + \alpha^{(3)}, \quad \gamma_2 = -\beta_2 - \beta_3 = \alpha^{(1)} - \alpha^{(3)},$$

and consider $P = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2 + \mathbb{Z}\gamma$. Since γ_1 , γ_2 and γ are mutually orthogonal, we have $V_P = V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2} \otimes V_{\mathbb{Z}\gamma}$. One can easily see that

$$P = \langle \alpha^{(1)} - \alpha^{(3)}, \alpha^{(2)} + 2\alpha^{(3)}, 6\alpha^{(3)} \rangle_{\mathbb{Z}}. \quad (5.9)$$

Hence we have a coset decomposition

$$L = \cup_{i=0}^5 (i\alpha^{(3)} + P).$$

Since $\alpha^{(3)} = \frac{1}{6}\gamma_1 - \frac{1}{2}\gamma_2 + \frac{1}{3}\gamma$, we have

$$3\alpha^{(3)} + P = \left(\frac{1}{2}\gamma_1 + \mathbb{Z}\gamma_1\right) + \left(\frac{1}{2}\gamma_2 + \mathbb{Z}\gamma_2\right) + \mathbb{Z}\gamma.$$

Therefore,

$$\begin{aligned} \text{Com}_{V_L}(V_{\mathbb{Z}\gamma}) &= \text{Com}_{V_P \oplus V_{3\alpha^{(3)}+P}}(V_{\mathbb{Z}\gamma}) \\ &= (V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}) \oplus (V_{\frac{1}{2}\gamma_1 + \mathbb{Z}\gamma_1} \otimes V_{\frac{1}{2}\gamma_2 + \mathbb{Z}\gamma_2}). \end{aligned} \quad (5.10)$$

- Lemma 5.3.** (1) *The eigenvalues for τ' on $V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}$ are ± 1 .*
(2) *The eigenvalues for τ' on $V_{\frac{1}{2}\gamma_1 + \mathbb{Z}\gamma_1} \otimes V_{\frac{1}{2}\gamma_2 + \mathbb{Z}\gamma_2}$ are $\pm\sqrt{-1}$.*
(3) $(V_P \oplus V_{3\alpha^{(3)}+P})^{\tau'} = (V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2})^{\tau'} \otimes V_{\mathbb{Z}\gamma}.$

Proof. Recall that we regard L as a subset of V_L . Under the canonical identification between V_L and $V_{A_1}^{\otimes 3}$, we have

$$\begin{aligned} \gamma_1 &= \alpha \otimes \mathbf{1} \otimes \mathbf{1} - 2(\mathbf{1} \otimes \alpha \otimes \mathbf{1}) + \mathbf{1} \otimes \mathbf{1} \otimes \alpha, \\ \gamma_2 &= \alpha \otimes \mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{1} \otimes \alpha, \\ \gamma &= \alpha \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \alpha \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \alpha, \\ e^{\pm\gamma_1} &= e^{\pm\alpha} \otimes e^{\mp 2\alpha} \otimes e^{\pm\alpha}, \\ e^{\pm\gamma_2} &= e^{\pm\alpha} \otimes \mathbf{1} \otimes e^{\mp\alpha}, \\ e^{\pm\gamma} &= e^{\pm\alpha} \otimes e^{\pm\alpha} \otimes e^{\pm\alpha}, \end{aligned}$$

respectively. Then by Lemma 5.2, we have

$$\tau'(\gamma_1) = \gamma_1, \quad \tau'(e^{\pm\gamma_1}) = -e^{\pm\gamma_1}, \quad (5.11)$$

$$\tau'(\gamma_2) = -\gamma_2, \quad \tau'(e^{\pm\gamma_2}) = -e^{\mp\gamma_2}, \quad (5.12)$$

$$\tau'(\gamma) = \gamma, \quad \tau'(e^{\pm\gamma}) = e^{\pm\gamma}. \quad (5.13)$$

Since $V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}$ is generated by the set $\{\gamma_1, \gamma_2, e^{\pm\gamma_1}, E^{\gamma_2}, F^{\gamma_2}\}$ consisting of eigenvectors for τ' whose eigenvalues are ± 1 , we have the assertion (1).

We note that $V_{\frac{1}{2}\gamma_1 + \mathbb{Z}\gamma_1} \otimes V_{\frac{1}{2}\gamma_2 + \mathbb{Z}\gamma_2}$ is an irreducible $V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}$ -module and so it is generated by a nonzero vector $u = e^{\frac{1}{2}(\gamma_1 + \gamma_2)} + \sqrt{-1}e^{\frac{1}{2}(\gamma_1 - \gamma_2)}$. Since

$$\begin{aligned} \tau'(e^{\frac{1}{2}(\gamma_1 + \gamma_2)}) &= \tau'(e^{\alpha^{(1)} - \alpha^{(2)}}) = e^{-\alpha^{(2)} + \alpha^{(3)}} = e^{\frac{1}{2}(\gamma_1 - \gamma_2)}, \\ \tau'(e^{\frac{1}{2}(\gamma_1 - \gamma_2)}) &= \tau'(e^{-\alpha^{(2)} + \alpha^{(3)}}) = -e^{\alpha^{(1)} - \alpha^{(2)}} = -e^{\frac{1}{2}(\gamma_1 + \gamma_2)} \end{aligned}$$

we have $\tau'(u) = \sqrt{-1}u$. Hence by (1), the assertion (2) holds.

The assertion (3) follows from (1), (2) and (5.13). \square

Here we note that $\langle \gamma_1, \gamma_1 \rangle = 12$ and $\langle \gamma_2, \gamma_2 \rangle = 4$. Let

$$g = I_{\frac{1}{24}\gamma_1} \otimes (I_{\frac{1}{8}\gamma_2} \theta_{\mathbb{Z}\gamma_2}) \in \text{Aut}(V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}).$$

Then by (5.11) and (5.12), the restriction of τ' to $V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}$ coincides with the automorphism g . Hence (5.10) and Lemma 5.3 imply the following proposition.

Proposition 5.4. *Let L and τ' be as above. Then*

$$\text{Com}_{V_L}(V_{\mathbb{Z}\gamma})^{\tau'} = (V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2})^g.$$

Let M and τ be as in Section 4. Then we see that

$$\begin{aligned} \rho(M^\tau) &= \text{Com}_{V_L^+}(V_{\mathbb{Z}\gamma})^{\tau'} \\ &= \text{Com}_{V_L}(V_{\mathbb{Z}\gamma})^{\langle \tau', \theta' \rangle} \\ &= (V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2})^{\langle g, \theta' \rangle} \end{aligned}$$

by the definition of τ' , Theorem 5.1 and Proposition 5.4, where

$$\theta' = \theta_{\mathbb{Z}\gamma_1} \otimes \theta_{\mathbb{Z}\gamma_2} \in \text{Aut}(V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}).$$

Let $G = \langle g, \theta' \rangle$ be a subgroup of $\text{Aut}(V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2})$ generated by g and θ' . We note that

$$G = \langle (I_{\frac{1}{24}\gamma_1} \theta_{\mathbb{Z}\gamma_1}) \otimes I_{\frac{1}{8}\gamma_2}, I_{\frac{1}{24}\gamma_1} \otimes (I_{\frac{1}{8}\gamma_2} \theta_{\mathbb{Z}\gamma_2}) \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Theorem 5.5. *Let $M = \text{Com}_{V_{A_1^4}}(\mathcal{L}(4, 0))$ and τ a cyclic permutation of $V_{A_1^4}$ of length 4 as in Section 4. Let G be as above. Then M^τ is isomorphic to the orbifold model $(V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2})^G$.*

Let

$$f = I_{\frac{1}{48}\gamma_1} \otimes I_{\frac{1}{16}\gamma_2} \in \text{Aut}(V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}).$$

By (3.1), we have

$$\begin{aligned} f^{-1}((I_{\frac{1}{24}\gamma_1} \theta_{\mathbb{Z}\gamma_1}) \otimes I_{\frac{1}{8}\gamma_2})f &= \theta_{\mathbb{Z}\gamma_1} \otimes I_{\frac{1}{8}\gamma_2}, \\ f^{-1}(I_{\frac{1}{24}\gamma_1} \otimes (I_{\frac{1}{8}\gamma_2} \theta_{\mathbb{Z}\gamma_2}))f &= I_{\frac{1}{24}\gamma_1} \otimes \theta_{\mathbb{Z}\gamma_2}. \end{aligned}$$

Let $g_1 = \theta_{\mathbb{Z}\gamma_1} \otimes I_{\frac{1}{8}\gamma_2}$ and $g_2 = I_{\frac{1}{24}\gamma_1} \otimes \theta_{\mathbb{Z}\gamma_2}$. Then $\langle g_1, g_2 \rangle = f^{-1}Gf \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and the following corollary holds.

Corollary 5.6. *The vertex operator algebra M^τ is isomorphic to the orbifold model $(V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2})^{\langle g_1, g_2 \rangle}$.*

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